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Systems
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Note on Degeneracy

by
Alamuru S. Krishna

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Department of Operations Research
Stanford University
Stanford, CA 94305

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SYSTEMS OPTIMIZATION LABORATORY
DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305-4022

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Note on degeneracy

By A.S.Krishna, Stanford University

Abstract

Given a linear program in standard form, $Min\ cx\ s.t.\ Ax = b,\ x \geq 0$, where A is an $m \times n$ matrix with rational coefficients, one technique used to resolve degeneracy in the simplex algorithm is the lexicographic rule. This rule adjoins to b a non-singular $m \times m$ square matrix M. The appended columns of M are updated along with b on each iteration. When the ratio test for determining the pivot row results in a tie, the ratio test is applied to the corresponding elements of the updated columns of M in turn, from left to right, until the tie is resolved. In this note we prove that it is only necessary instead, to adjoin to b a single column d whose ith component is $d_i = \pi^i$, (or preferably the fractional part of π^i in order that $|d_i| < 1$). Any transcendental number~~, like $e = 2.73\ldots$,~~ the base of the ~~natural logirthm,~~ can be used instead of $\pi = 3.14\ldots$. The proof exploits the fundamental property of a transcendental number namely, it can never be a root of a polynomial equation $\alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_p x^p = 0$ when $\alpha_1, \alpha_2, \ldots, \alpha_p$ are rational and not all zero.

Review of standard ϵ – perturbation method or lexicographic rule :

In ϵ-perturbation method we perturb the RHS to $(b_1 + \epsilon, b_2 + \epsilon^2, \ldots, b_m + \epsilon^m)$. At any iteraton the right hand side is $(\bar{b}_1 + \bar{p}_1(\epsilon), \ldots, \bar{b}_m + \bar{p}_m(\epsilon))$ where $\bar{p}_i$ is the polynomial $\sum_{k=1}^{m} \beta_{ik}\epsilon^k$ where β stands for the inverse of the basic matrix in the curent iteration. Given the variable entering basis, it can be proven that $\exists\ \epsilon_0 > 0$ such that $\forall\ 0 < \epsilon < \epsilon_0$, the variable that leaves the basis is uniquely determined. Moreover this lexicographic rule for choosing the entering basic variable solves the original problem in finitely many iterations. The key point in the proof is based on the observation that at any iteration the polynomials $\sum_{k=1}^{m} \beta_{ik}\epsilon^k\ for\ i = 1, \ldots, m$ are independent, in the sense that no non-zero linear combination of them is identically equal to zero. This implies that for any choice of $0 < \epsilon < \epsilon_0$, no two rows can be tied.

However perturbing in this way requires updating the full matrix $M^{-1} = [\beta_{ij}]$ and possibly making many comparisons since, in the worst case, we might have to compare elements of m additional columns. However, when all entries in the initial tableau are rational, we propose a different perturbation scheme which requires the updating and comparisons in case of ties in exactly one additional column. Moreover this column does not depend on the values of A or b and is the same for all problems with m rows.

Towards that end let us suppose that all entries of the initial tableau are rational. We perturb the RHS to $(b_1 + d_1\epsilon, \ldots, b_m + d_m\epsilon)$ where $d_1, \ldots, d_m$ will be specified later. At any subsequent iteration t, let the updated RHS be $\overline{b}_1 + \overline{d}_1\epsilon, \overline{b}_2 + \overline{d}_2\epsilon, \ldots, \overline{b}_m + \overline{d}_m\epsilon$. The values of $\overline{b}_i$ and $\overline{d}_i$ can be found by standard pivot updating or derived from the original tableau by the formula $\overline{b}_i = \sum_k \beta_{ik} b_k$ and $\overline{d}_i = \sum_k \beta_{ik} d_k$ for $i = 1, \ldots, m$ where, as usual, β stands for the inverse of the current basis matrix.

We want to choose the coefficients $d = (d_1, \ldots, d_m)$ so that for any subsequent iteration t, either $\overline{b}_i > 0$ or $\overline{b}_i = 0$ and $\overline{d}_i > 0$. To find such a d, we note that if A consists of all rational entries, then all subsequent basis inverse elements β_{ij} are guaranteed to be rational. Hence it is enough to choose $d_1, \ldots, d_m$ s.t. $\sum_k \alpha_k d_k \neq 0$ whenever all α_k are rational except if $\alpha_k = 0$ for all k . This is easily accomplished by choosing $d_k = d^k$ where d is any transcendental number, say, $\pi = 3.14156\ldots$ for example. In that case the requisite property is satisfied by any transcendental number, indeed this is the definition of a transcendental number.

To summarize, the following perturbation scheme involves less updating and smaller possible number of comparisions than the conventional lexicographic scheme and guarantees, in the case of a rational problem, finite termination.

Apply the simplex method with the following choice of the variable leaving the basis. First append to the RHS the column $d = (\pi, \pi^2, \ldots, \pi^m)^T$ or alternatively $d = (d_1, \ldots, d_m)$ where d_i is the fractional part of π^i. For some subsequent iteration let $\overline{A}, \overline{b}, \overline{c}, \overline{d}$ denote the elements of the current tableau where $\overline{d}$ is the updated column d which is appended. Then the steps are:

STEP 1: $s = \operatorname{argmin}_{j} (\overline{c}_j)$

STEP 2: If $\overline{c}_s \geq 0$, terminate.
(current basic feasible solution is optimal)

STEP 3: Denote by $R = \{i | \overline{a}_{is} > 0\}$.
If R is empty, terminate.
obj(current basic solution $+\lambda$ homogeneous solution)$\rightarrow -\infty$
as $\lambda \rightarrow +\infty$

STEP 4: Denote by $\overline{R} = \{k \in R | \overline{b}_k / \overline{a}_{ks} \leq \overline{b}_i / \overline{a}_{is} \ \forall\, i \in R\}$
$r = \underset{k \in \overline{R}}{\operatorname{argmin}} (\overline{d}_k / \overline{a}_{ks})$

STEP 5: Update tableau by pivoting on $\overline{a}_{rs}$.

STEP 6: GO TO STEP 1.

The above proof is valid if the coefficients are rational and the computations are carried in exact arithmetic. As for the case of irrational coefficients the following observation can be made: Given any table, there are only finitely many tables which can be generated from it by pivoting. So altogether there are only countably many tables that might show up at some point. So perturbing the RHS to $(b_1+d\epsilon, \ldots, b_m+d^m\epsilon)$, we get countably many polynomials in d that might show up as coefficients of ϵ in RHS. So surely $\exists\ d_0$ which is not a root of any of these polynomials. Hence choosing such a $d = d_0$ can never result in a tie for the choice of pivot row since, having a tie at some iteration implies having 0 in the next iteration, which we have just ruled out.

For the case of irrational coefficients all this is not of much practical significance, since there appears to be no practical way of finding such an d_0. Moreover we need not worry too much about this case in practice since all practical problems are rational, i.e., only rationals can be represented on a computer. This same criticism applies in the rational case as well since, from a practical point of view, neither can a transcendental like π or e be represented on computer.

Theoretically we have finite termination with, say, π. Therefore we have finitely many polynomails in d as coefficients of ϵ for which π was not a root. Hence no number in a small neighbourhood of π can be root for those polynomials either. So representing π sufficiently accurately does the trick, though once again from a practical point of view we can not determine *a priori* how much accuracy is sufficient and whether requiring that much accuracy is worthwhile or not.

Another point of practical concern is computing the powers of π. If m is large it is, of course, not advisable to compute and store π^m in a computer. Instead we can still have rational independence of polynomials as explained above if we choose our $d_1, d_2, \ldots, d_m$ as follows. Define $d_0 = 1$, then $d_i = \pi d_{i-1} - [\pi d_{i-1}]$ for $i = 1, \ldots, m$ where $[\pi d_{i-1}]$ denotes the greatest integer less than or equal to πd_{i-1}. In a computer π is replaced by $\overline{\pi}$, the first p significant digits of π; moreover d_i is replaced by $\overline{d_i}$, the first q significant digits of d_i. It is therefore possible, due to rounding, that $d_i = d_{i+k}$ for some i and $k > 0$. If this is the case the sequence of d_i will cyclically repeat. To test this possibility we generated $\{d_i\}$ for $i = 1, \ldots, 5000$, where π was replaced by $\bar{\pi}$, the first 16 digits of π and all the arithmetic was carried to 16 decimal places. No repetition was found in the sequence.

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